

ON REGULARIZATION OF J -PLURISUBHARMONIC FUNCTIONS

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ABSTRACT. We show that on almost complex surfaces plurisubharmonic functions can be locally approximated by smooth plurisubharmonic functions. The main tool is the Poletsky type theorem due to U. Kuzman.

1. INTRODUCTION

Let (M, J) be an almost complex manifold. In his paper [H] Hagui defines plurisubharmonic functions on M as upper semicontinuous functions which are subharmonic on every J -holomorphic disk. Recently Harvey and Lawson proved that a locally integrable function u is plurisubharmonic iff a current $i\partial\bar{\partial}u$ is positive (see [H-L]).

It is a very natural open question in this theory whether any plurisubharmonic function is (locally) a limit of a decreasing sequence of smooth plurisubharmonic functions. The Richberg type theorem was proved in [P]. This gives a positive answer in a case of continuous functions. In this note we prove it for all plurisubharmonic functions in the (complex) dimension¹ 2.

Theorem 1. *Let $\dim M = 2$ and $P \in M$. Then there is a domain D which is a neighbourhood of P such that for every $u \in \mathcal{PSH}(D)$ there exists a decreasing sequence $\psi_k \in C^\infty \cap \mathcal{PSH}(D)$ such that $\psi_k \rightarrow u$.*

As an immediate consequence of Theorem 1 and proposition 5.2 from [P] we obtain the following

Corollary 2. *Let $\dim M = 2$ and u, v in $W_{loc}^{1,2} \cap \mathcal{PSH}(M)$. Then a current $i\partial\bar{\partial}u \wedge i\partial\bar{\partial}v$ defined in [P] is a (positive) measure.*

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¹In this note by the dimension of an almost complex manifold we mean the complex dimension which is a half of the real dimension.

In particular the Monge-Ampère operator $(i\partial\bar{\partial}u)^2$ is well defined for any bounded psh function u on an almost complex surface (compare proposition 4.2 in [P]). On domains in \mathbb{C}^2 it was proved by Błocki (see [B]) that a set $W_{loc}^{1,2} \cap \mathcal{PSH}$ is a natural domain for the Monge-Ampère operator.

2. PROOF

2.1. J -holomorphic discs. A good reference for the (local) theory of J -holomorphic discs is [I-R]. In this subsection J is \mathcal{C}^1 close to J_{st} (in particular $(J + J_{st})$ is invertible) where J_{st} is the standard (integrable) almost complex structure in \mathbb{C}^n . Let \mathbb{D} be a unit disc in \mathbb{C} . A function $u : \mathbb{D} \rightarrow (\mathbb{C}^n, J)$ is J -holomorphic if and only if

$$\frac{\partial u}{\partial \bar{z}} + Q(u) \frac{\partial u}{\partial z} = 0$$

when

$$Q = (J - J_{st})(J + J_{st})^{-1}.$$

Let $0 < \alpha < 1$ and $T : \mathcal{C}^{0,\alpha}(\bar{\mathbb{D}}, \mathbb{C}^n) \rightarrow \mathcal{C}^{1,\alpha}(\bar{\mathbb{D}}, \mathbb{C}^n)$ be the Cauchy-Green operator given by

$$Tu = \frac{1}{\pi} \int_{\mathbb{D}} \frac{u(\zeta)}{\cdot - \zeta} d\zeta.$$

Set

$$\Phi u = u + T(Q(u) \frac{\partial u}{\partial z})$$

and

$$\Psi u = \Phi u + (u - \Phi u)(0).$$

By the definition $\Psi u(0) = u(0)$. Note that $u \in \mathcal{C}^{1,\alpha}(\bar{\mathbb{D}}, \mathbb{C}^n)$ is J -holomorphic in \mathbb{D} iff $\Phi(u)$ is J_{st} -holomorphic. Because $d\Psi$ is close to Id , the map $\Psi : \mathcal{C}^{1,\alpha}(\bar{\mathbb{D}}, \mathbb{C}^n) \rightarrow \mathcal{C}^{1,\alpha}(\bar{\mathbb{D}}, \mathbb{C}^n)$ is a local diffeomorphism and there is a constant C_0 such that $\|(d\Psi)^{-1}\| \leq C_0$ everywhere.

We will use the following

Lemma 3. *Let $V \in \mathbb{C}^n$. For any $u \in \mathcal{C}^{1,\alpha}(\bar{\mathbb{D}}, \mathbb{C}^n)$ there is $v \in \mathcal{C}^{1,\alpha}(\bar{\mathbb{D}}, \mathbb{C}^n)$ such that $\Psi(v) = \Psi(u) + V$ and $\|u - v\|_{\mathcal{C}^{1,\alpha}} \leq C_0|V|$.*

Proof: Set $U_t = \Psi(u) + tV$ and

$$S = \{t \in [0, 1] : \exists w \in \mathcal{C}^{1,\alpha}(\bar{\mathbb{D}}, \mathbb{C}^n) \text{ s. t. } \Psi(w) = U_t, \|u - w\|_{\mathcal{C}^{1,\alpha}} \leq tC_0|V|\}.$$

S is nonempty, by the inverse function theorem it is open, by the Arzelà-Ascoli theorem it is closed and hence $S = [0, 1]$. \square

2.2. Disc envelope. Let $p \in \Omega \subset M$ and let $\mathcal{O}_p(\bar{\mathbb{D}}, \Omega)$ be a set of J -holomorphic discs $\lambda : \bar{\mathbb{D}} \rightarrow \Omega$ with $\lambda(0) = p$. For an upper semicontinuous function $f : \Omega \rightarrow \mathbb{R}$ we consider the following disc envelope:

$$P_\Omega f(p) = \inf_{\lambda \in \mathcal{O}_p(\bar{\mathbb{D}}, \Omega)} \frac{1}{2\pi} \int_0^{2\pi} f \circ \lambda(e^{it}) dt.$$

We need the following Lemma.

Lemma 4. *Let $\Omega_1 \Subset \Omega_2 \subset \mathbb{C}^n$ and let J be an almost complex structure on Ω_2 which is \mathcal{C}^1 close to J_{st} . Let $f \in \mathcal{C}(\Omega_2)$ be such that*

$$P_{\Omega_1} f = (P_{\Omega_2} f)|_{\Omega_1}.$$

Then $P_{\Omega_1} f \in \mathcal{C}(\Omega_1)$.

Proof: We can assume that f is uniformly continuous on Ω_2 with a modulus of continuity ω and J is \mathcal{C}^1 close to J_{st} on \mathbb{C}^n . Set any $0 < \delta < C_0^{-1} \text{dist}(\partial\Omega_1, \partial\Omega_2)$. Let $\varepsilon > 0$, and $p, q \in \Omega_1$ with $|p - q| \leq \delta$. There is $\lambda \in \mathcal{O}_p(\bar{\mathbb{D}}, \Omega_1)$ such that:

$$P_{\Omega_1} f(p) \geq \frac{1}{2\pi} \int_0^{2\pi} f \circ \lambda(e^{it}) dt - \varepsilon.$$

By Lemma 3 there is a function $\mu \in \mathcal{C}^{1,\alpha}(\bar{\mathbb{D}}, \mathbb{C}^n)$ such that $\Phi(\mu) = \Phi(\lambda) + w - z$ and $\|\lambda - \mu\|_{L^\infty} \leq C_0|p - q|$. Since functions $(z \mapsto \mu(rz))$ are in $\mathcal{O}_q(\bar{\mathbb{D}}, \Omega_2)$ for $1 > r > 0$ we can estimate

$$\begin{aligned} P_{\Omega_1} f(q) &= P_{\Omega_2} f(q) \leq \frac{1}{2\pi} \int_0^{2\pi} f \circ \mu(e^{it}) dt \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} (f \circ \lambda(e^{it}) + \omega(C_0\delta)) dt \leq P_{\Omega_1} f(p) + \omega(C_0\delta) + \varepsilon. \end{aligned}$$

Letting ε to 0 we can conclude that $P_{\Omega_1} f$ is uniformly continuous with a modulus of continuity $\tilde{\omega}(x) = \omega(C_0x)$. \square

2.3. Kuzman-Poletsky theorem. For a domain $\Omega \subset M = \mathbb{C}^n$ and an upper semicontinuous function f , Poletsky (see [Po]) proved that $H_\Omega f$ is a plurisubharmonic function (moreover it is the largest plurisubharmonic minorant of f). The key tool in the proof of Theorem 1 is a result of Kuzman, who showed the same for any 2-dimensional almost complex manifold (see theorem 1 in [K]). The only reason for the assumption about a dimension in our Theorem is just this assumption in Kuzman's theorem.

Proof of Theorem 1: The theorem is local hence we can assume that $P \in \mathbb{C}^2$ and J is \mathcal{C}^1 close to J_{st} . We can choose a neighbourhood D of P such that there exists a positive continuous strictly J -plurisubharmonic² exhaustion function ρ on D .³ Set $u \in \mathcal{PSH}(D)$. Let us take a decreasing sequence of continuous functions ϕ_k tending to u . We can modify ρ such that $\lim_{z \rightarrow \partial D} (\rho - \phi_1) = +\infty$ and put $\tilde{\phi}_k = \max\{\phi_k, \rho - k\}$. There are domains $D_k \Subset D$ such that $\tilde{\phi}_k = \rho - k$ on some neighbourhood U_k of $D \setminus D_k$. By Kuzman's result $\hat{\phi}_k = P_D \tilde{\phi}_k, P_{D_k} \tilde{\phi}_k$ are J -plurisubharmonic. Note that $\hat{\phi}_k = \rho - k$ on U_k and $P_{D_k} \tilde{\phi}_k = \rho - k$ on $D_k \cap U_k$, hence by Lemma 4 $\hat{\phi}_k \in \mathcal{C}(D)$. Thus we get a decreasing sequence of continuous J -plurisubharmonic functions $\hat{\phi}_k$ tending to u .

By the Richberg theorem (see theorem 3.1 in [P]) there are functions $\psi_k \in \mathcal{C}^\infty \cap \mathcal{PSH}(D)$ such that

$$\hat{\phi}_k + 2^{-k-1}\rho \leq \psi_k \leq \hat{\phi}_k + 2^{-k}\rho$$

and we can see that a sequence ψ_k decreases to u . \square

2.4. Final remarks. Now we formulate two open problems related to the regularisation problem.

Question 1. Is it possible to extend (at least locally) the Poletsky theorem to almost complex manifolds in the dimension larger than 2?

Question 2. For a given (almost Stain) manifold M , and any continuous function f on M is the largest plurisubharmonic minorant \hat{f} of f continuous?

From the proof above it is clear that a positive answer to first question give (locally) a positive answer to second question.

Obviously positive answer to second question gives us the approximation of any plurisubharmonic function by continuous plurisubharmonic functions on M (in particular, surprisingly for the author, \hat{f} can be discontinuous even in some domains in \mathbb{C}^n). If moreover M admits a continuous strictly plurisubharmonic function for any plurisubharmonic function u , then there is a decreasing sequence of smooth plurisubharmonic functions ψ_k tending to u .

²A function u is strictly plurisubharmonic on D means as usually that for any $\varphi \in \mathcal{C}_0^2$ there is $\varepsilon > 0$ such that $u + \varepsilon\varphi$ is plurisubharmonic. We write here J -plurisubharmonic instead of plurisubharmonic to stress that a function is plurisubharmonic with respect to the almost complex structure J (note that on D we have also the almost complex structure J_{st}).

³Such domain D is called a Stain domain, see [D-S]. Here we can take $D = \{|z - P| < \varepsilon\}$ for $\varepsilon > 0$ small enough.

Actually, for any $u \in \mathcal{PSH}$ there is a decreasing sequence of continuous J -plurisubharmonic functions tending to u iff $\hat{f} \in \mathcal{C}(M)$ for any $f \in \mathcal{C}(M)$. Indeed, let $K \Subset M$ and ψ_k be a decreasing sequence of continuous J -plurisubharmonic functions tending to \hat{f} . For any $\varepsilon > 0$ there is k such that $\psi_k \leq f + \varepsilon$ on K and hence

$$\psi_k - \varepsilon \leq \hat{f} \leq \psi_k.$$

We can conclude that the convergence is uniform on compact sets and \hat{f} is continuous.

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